Thermodynamics

Basic definitions:

Heat: The energy transferred between two or more systems or surroundings as a result of temperature difference only, is called heat

Unit : joule in SI, Calorie in CGS system.

Temperature: Temperature is defined as the thermal condition of a body which determines its ability to describe the relative hotness or coldness of a body.

Unit: Kelvin(K) in absolute scale.

System: The region ay contains a collection of large number of atoms or molecule with in a real or imaginary surroundings

Ex: A gas enclosed in a cylinder having movable piston.

Surroundings: The medium or matter or vacuum that surrounds the system which may participate in the process of exchange of matter or energy or both with the system.

Types of systems: There are three types of systems viz.,

a) Closed System b) Open System c) Isolated system.

Closed System: The closed system is the one which exchange only energy with its surroundings, but no exchange of matter with the surroundings.

Ex: Cylinder fitter with a movable piston.

Open System: An open system is the one which can exchange energy as well as matter with the surroundings.

Ex: An open vessel / Air compressor.

Isolated system: An isolated system is one which cannot exchange energy or matter with its surroundings.

Ex: The fluid contained by a thermos flask..

Thermodynamic equilibrium: If a system is simultaneously in a state of mechanical equilibrium, chemical equilibrium and thermal equilibrium, then the system is said to be in thermodynamic equilibrium.

Internal energy of a system (u): The internal energy of a system is defined as the sum of all the energies contained in the system.

The internal energy of a gas is sum of kinetic energy and intermolecular potential energies of the gas.

Internal energy of the gas

$U = \Sigma PE + \Sigma KE$

ΣPE arises due to the intermolecular forces of attraction/repulsion among the molecules.

ΣKE arises due to the translation, vibration and rotational kinetic energies of the molecules of the gas.

First law of thermodynamics

Statement: If the quantity of heat supplied to a system is capable of doing work, then the quantity of heat absorbed by the system is equal to the sum of the increase in the internal energy and the external work done by it.

$$\boxed{dQ = dU + dW} \quad \rightarrow (1)$$

Where dQ → The quantity of heat energy supplied to the system.

dU→ The increase in internal energy.

dW→ The external work done by the system.

Sign convention:

(1) If the heat energy dQ is added to the system is taken as positive.
(2) If the heat energy dQ is released from the system is taken as negative
(3) The increase in internal energy dU is taken as positive.
(4) The decrease in internal energy dU is taken as negative.
(5) The work is done dW by the system is taken as positive.
(6) The work is done dW on the system is taken as negative.

Significance:

(1) This law verifies the law of conservation of energy in thermodynamics.

(2) This law introduces the concept of internal energy.

(3) This law is applicable for all the states and natural process.

Limitations:

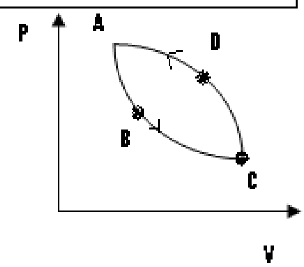
(1) This law fails to explain the direction of heat flow.

(2) This law fails to explain the concept of entropy.

<u>Thermodynamic processes:</u>

Path:

The locus of the series of points representing the states through which the system passes is called path.



Consider a system undergoes a change of state from state A to State C as shown in figure passing through intermediate state by absorbing different amount of heat along ABC is the path for the system from state A to State C.

Process:

The complete description of change of state along with the path is called a process.

It is observed that a system undergoes a process along the path represented by the graph ABC by absorbing some amount of heat. If the system is made to undergo another change of state again from CAD by radiating some amount of heat and that the complete description of the change of state is called process.

Reversible process:

A reversible process is that which can be retraced in the opposite direction i.e. if a small change is made in the forward direction, the process is reversed completely and the working substance passes through exactly the same directions as it does in the direct process.

Example: -Consider the conversion of water at 0^0C into ice at 0^0C by removing certain quantity of heat. If the same quantity of heat is supplied to 0^0C ice it will melt and will be converted to its original state of water.

Irreversible process:

A process is which reversal does not take place is called as irreversible process.

Example:-The conduction of heat from hot body to a cold body is an example of irreversible process.

Isothermal process:-

A process in which a system undergoes physical changes in such a way that the temperature remains constant by the exchange of heat energy with the surroundings is known as isothermal process.

(i) In this process Temperature remains constant hence $\Delta T = 0$

(ii) $P \propto 1/V$ or PV = constant i.e. P-V graph is a rectangular hyperbola with $P_1V_1=P_2V_2$

(iii) Since $\Delta T = 0$, hence $\Delta U = 0$ for an ideal gas where U is internal energy is a function of temperature.

Work done during Isothermal process:

Consider one mole of an ideal gas enclosed in an isothermal chamber i.e. the walls of chamber; base and piston are good conductors. Now the gas expands at constant temperature isothermally from volume V_1 to V_2. Let the corresponding pressure be P_1 and P_2

At any instant let "P" be the pressure of the gas, the motion of the piston through an elementary change dx, area of the piston is "A" then the work-done is given by

$dW = \text{Force} \times dx = PAdx = PdV$

Where dV is the infinitesimally small increase in volume of the gas at that pressure.

$$W = \int dW = \int_{V_1}^{V_2} P.dV$$

But from gas equation $P = RT/V$

$$\therefore\ W = \int_{V_1}^{V_2} \frac{RT}{V}.dV$$

Or $$W = RT \log_e\left[\frac{V_2}{V_1}\right]$$

Or $$\boxed{W = 2.303\ RT \log_{10}\left[\frac{V_2}{V_1}\right]}$$

The expression for "n" moles the work done is

$$W = 2.303nRT \log_{10}\left[\frac{V_2}{V_1}\right]$$

Adiabatic Process:

A process in which a system undergoes physical changes in such a way that the total heat energy remains constant hence heat energy neither allowed to enter the system from the surrounding nor allowed to leave the system to the surroundings is called an adiabatic process.

(i) In adiabatic process, the total heat energy remains constant hence $\Delta Q = 0$

(ii) From the first law of thermodynamics $dU + dW = 0$ or $dU = -dW$.

(iii) In this process PV^{γ} = constant (or) $TV^{\gamma-1}$ = constant (or) $P^{1-\gamma}T^{\gamma}$ = constant

Work done during adiabatic process:

Consider one mole of an ideal gas enclosed in an adiabatic chamber i.e. the walls of chamber; base and piston are bad conductors. Now the gas expands at constant heat energy from volume V_1 to V_2. Let the corresponding pressure be P_1 and P_2

At any instant let "P" be the pressure of the gas, the motion of the piston through an elementary change dx, area of the piston is "A" then the work-done is given by

$dW = \text{Force} \times dx = PAdx = PdV$

Where dV is the infinitesimally small increase in volume of the gas at that pressure.

$$W = \int dW = \int_{V_1}^{V_2} P.dV$$

For an adiabatic change

$PV^{\gamma} = K$ (constant) $\therefore P = K/V^{\gamma}$

$$\therefore W = \int_{V_1}^{V_2} \frac{K}{V^{\gamma}}.dV = K\left[\frac{V^{1-\gamma}}{1-\gamma}\right]_{V_1}^{V_2}$$

$$\therefore W = \frac{K}{1-\gamma}\left[V_2^{1-\gamma} - V_1^{1-\gamma}\right]$$

$$\text{(Or)}\ W = \frac{1}{1-\gamma}\left[KV_2^{1-\gamma} - KV_1^{1-\gamma}\right]$$

But $P_1V^{\gamma}{}_1 = P_2V^{\gamma}{}_2 = K$

$$\therefore W = \frac{1}{1-\gamma}\left[P_2V_2^{\gamma}V_2^{1-\gamma} - P_1V_1^{\gamma}V_1^{1-\gamma}\right]$$

$$W = \frac{1}{1-\gamma}[P_2V_2 - P_1V_1]$$

$$\therefore W = \frac{1}{1-\gamma}[RT_2 - RT_1]$$

$$\therefore W = \frac{R}{1-\gamma}[T_2 - T_1]$$

$$\boxed{\therefore W = \frac{R}{\gamma-1}[T_1 - T_2]}$$

This expression gives the work-done for a 1 mole of gas.

For ‘n’ moles of gas

$$\therefore W = \frac{nR}{\gamma-1}[T_1 - T_2]$$

Isobaric process:-

> The process in which a system undergoes a change in volume and temperature at constant pressure by the exchange of heat energy with the surroundings is called isobaric process

In this process the pressure remains constant.

Isochoric process:-

> The process in which a system undergoes a change in pressure and temperature at constant volume by the exchange of heat energy with the surroundings is called isochoric process

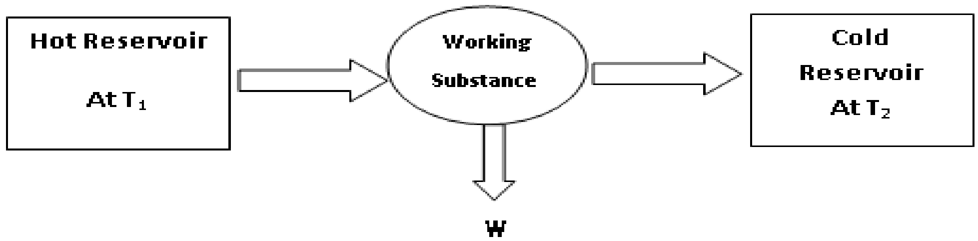
In this process the volume remains constant.

Heat Engine

The heat engine is a device used to convert heat energy into mechanical work through a medium, called working substance which is normally in the form of a vapour or gas.

A heat engine consists of the following parts

1. ***Source:*** It is a hot reservoir or a body which is at higher temperature T_1 and can extract any quantity of heat Q_1 without any change in its temperature.
2. ***Sink:*** It is a cold reservoir or body at lower temperature T_2 which can take any amount of heat Q_2 rejected by the working substance without any change in its temperature.
3. ***Working substance:*** It is substance depends on the engine which absorbs certain amount of heat from the source, converts a part of it into work and rejects the remaining heat to the sink.



Note: The efficiency of an engine is the ratio between work done (W) by the engine and the amount of heat absorbed (Q_1) by the engine

Efficiency of heat engine

$$\eta = \frac{W}{Q_1} = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1}$$

Carnot's engine and Carnot's cycle:-

Carnot devised an ideal engine which is based on a reversible cycle called Carnot's cycle.

A reversible heat engine operating between two temperatures is called a Carnot's engine.

Carnot's engine consists of

(i) A cylinder of perfectly non-conducting wall and perfectly conducting bottom. On the top of the cylinder is covered with piston made with adiabatic material and the movement is frictionless. The working substance is in the cylinder is assumed to be air which supposed to behave like a perfect gas.

(ii) A source of heat to supply heat at constant temperature say T_1

(iii) A sink or cold body to receive the rejected heat at a constant temperature say T_2

(iv) An insulating stand which is connected to the bottom of the cylinder.

Working:

The working substance in a Carnot's engine is taken through a reversible cycle consisting of the following operations in succession

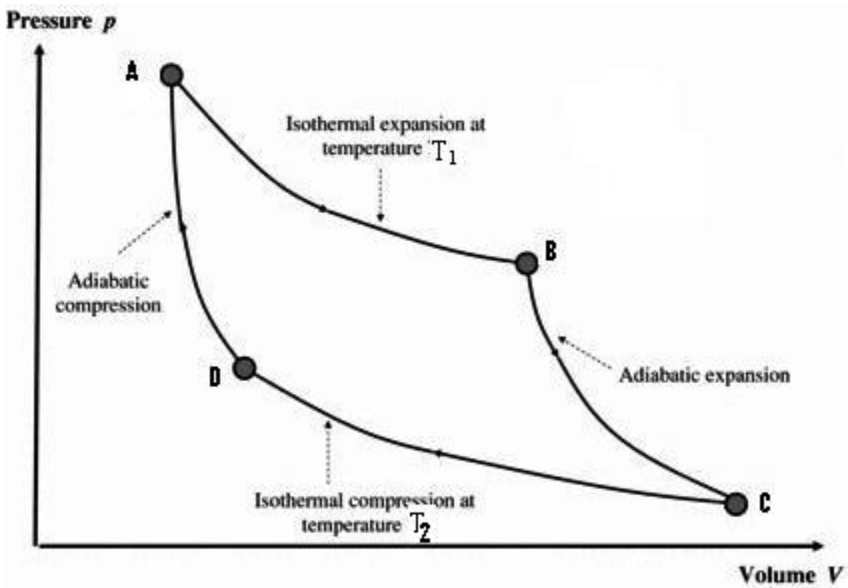
a) Isothermal Expansion b) Adiabatic Expansion c) Isothermal Compression

d) Adiabatic Compression

Isothermal expansion: The working substance containing "n" moles of ideal gas placed on source and the gas is allowed to expand slowly at constant temperature T_1 absorbing heat Q_1. This isothermal expansion is represented by the curve AB in the indicator diagram.

The state of gas at $A(P_1,V_1,T_1)$ changes to $B(P_2,V_2,T_1)$. The heat energy absorbed Q_1 equal to work-done by the gas

$Q_1 = W_1 = nRT_1 \log(V_2/V_1)$ →(1)



Adiabatic expansion: The working substance is then placed on the insulating stand and the gas is allowed to expand adiabatically till the temperature falls from T_1 to T_2, as shown in curve along BC

The state of gas changes from $B(P_2,V_2,T_1)$ to $C(P_3,V_3,T_2)$ the work done by the gas is

$$W_2 = \frac{nR(T_2-T_1)}{1-\gamma} \quad \rightarrow(2)$$

Isothermal compression: The working is now placed on the sing and the gas is compressed at constant temperature T_2 along the path CD by transferring a certain quantity of heat Q_2 to the sink

The state of gas changes from $C(P_3,V_3,T_2)$ to $D(P_4,V_4,T_2)$ the work done by the gas is

$$W_3 = nRT_2 \log(V_4/V_3) \quad \rightarrow(3)$$

Adiabatic compression: The working substance finally placed on the insulating stand and the compression along the curve DA

The state of gas changes from $D(P_4,V_4,T_2)$ to $A(P_1,V_1,T_1)$ the work done by the gas is

$$W_2 = \frac{nR(T_1-T_2)}{1-\gamma} \quad \rightarrow(4)$$

The total work-done by the by the gas $W = W_1 + W_2 + W_3 + W_4$

$$W = nRT_1\log(V_2/V_1) + \cancel{\frac{nR(T_2-T_1)}{1-\gamma}} + nRT_2\log(V_4/V_3) + \cancel{\frac{nR(T_1-T_2)}{1-\gamma}} \rightarrow (5)$$

Hence

$$W = nRT_1\log(V_2/V_1) + nRT_2\log(V_4/V_3) \rightarrow (6)$$

But from adiabatic relations $TV^{\gamma-1}$ = constant

Along BC $T_1V_2^{\gamma-1} = T_2V_3^{\gamma-1} \rightarrow (7)$

Along DA $T_1V_1^{\gamma-1} = T_2V_4^{\gamma-1} \rightarrow (8)$

By dividing above equations

$$V_2/V_1 = V_3/V_4 \rightarrow (9)$$

Hence equation (6) transform to

$$W = nR\log(V_2/V_1)(T_1-T_2)$$

The efficiency of Carnot's engine is given by the relation

$$\eta = W/Q_1 = \frac{nR\log\left(\frac{V_2}{V_1}\right)(T_1-T_2)}{nR\log\left(\frac{V_2}{V_1}\right)T_1} \rightarrow (10)$$

$$\boxed{\therefore\ \eta = 1-(Q_2/Q_1) = 1-(T_2/T1)}$$

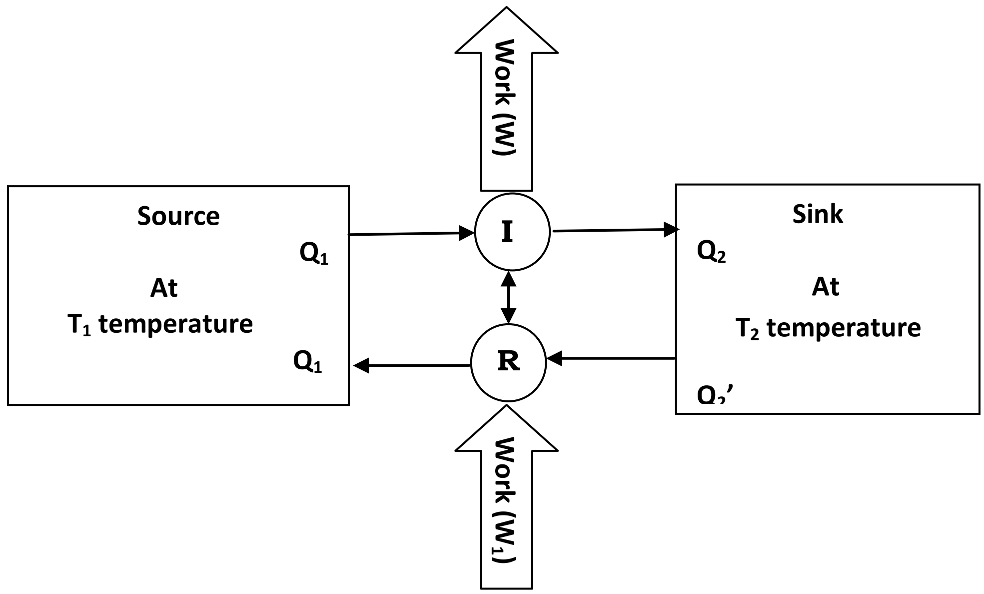
Carnot's Theorem:

Statement:

No heat engine can have more efficiency than the reversible engine operating between the same source(T_1) and sink (T_2) and the efficiency of Carnot engine is independent of the nature of the working substance.

Proof:

Consider an irreversible engine (I) and a reversible engine(R) working between same source and sink as shown in figure



Let the engine I extracts Q_1 heat from source at T_1 and releases Q_2 heat to sink at T_2

The work done by the engine $W = Q_1 - Q_2$

Heat given to the sink $Q_2 = Q_1 - W$ → (1)

The efficiency $\eta_I = W/Q_1$ → (2)

Now the reversible engine R works as refrigerator

This extracts Q_2' from sink at T_2 and releases Q_1 heat to the source at T_1 by absorbing work W_1 from surroundings

The work done on the system $W_1 = Q_1 - Q_2'$

Heat absorbed from sink $Q_2' = Q_1 - W_1$ → (3)

The efficiency $\eta_R = W_1/Q_1$ → (4)

Let the efficiency of I > the efficiency of R

$$\therefore \eta_I > \eta_R \Rightarrow W/Q_1 > W_1/Q_1 \Rightarrow W > W_1$$

By coupling these two engines such that work done by the Irreversible engine can be given to Reversible engine such that the compound engine extracts Q_2' from the sink and releases Q_2 to sink and given by

$$Q_2' - Q_2 = (Q_1 - W_1) - (Q_1 - W) = W - W_1 > 0 \because W > W_1$$

Hence the net heat taken from the sink is positive. But this is impossible for self acting machine extracts heat from cold body without aid of external agency.

$\therefore$ Our assumption that the irreversible engine is more efficient than the reversible engine is wrong.

Thus no heat engine working between a Source and sink can be more efficient than a reversible engine.

Since efficiency of engine is given by $\eta = 1 - T_2/T_1$ which is independent of nature of working substance.

Let us consider two reversible engines A and B working between the same source and sink then A cannot be more efficient than B and B cannot be more efficient than A so two heat engines are equally efficient.

Second law of thermodynamics:

Second law of thermodynamics is a fundamental law of nature which explains that heat can flow only from hot body to cold body by itself. There are several statements of this law. Two are the most significant viz.,

(a) Kelvin- Planck statement:

No process is possible whose sole result is the absorption of heat from a source and the complete conversion of the heat into work.

It is impossible to get a continuous supply of work from a body by cooling it to a temperature lower than that of the surroundings.

(b) Clausius statement:

No process is possible whose sole result is the transfer of heat from a colder object to a hotter object.

It is impossible for self acting machine to transfer heat from colder body to hotter body without an aid of external agency.

Concept of entropy:

The concept of entropy refers to state of order represented by S

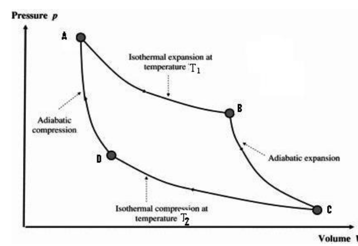
A change in order is a change in the number of ways of arranging the particles, and it is a key factor in determining the direction of any process

Solid ⟶ Liquid ⟶ Gas

More Order ---------------> Less Order

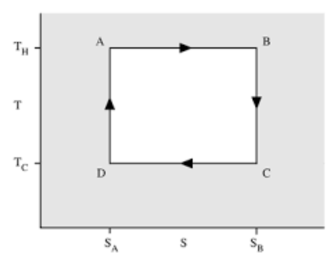
The increase in entropy is given by $dS = \frac{dQ}{T}$

Change in Entropy in a reversible and irreversible process:



Consider a reversible process such as Carnot cycle ABCDA as shown in the fig., during isothermal expansion from A→B, the temperature constant but the heat energy increases by Q_1, hence the entropy of the working substance from A→ B will be increases as Q_1/T_1

Since B→ C and D→A are adiabatic processes hence there is no change in entropy.
Along C→D the working substance rejects heat energy Q_2, hence the loss in entropy is Q_2/T_2.



Hence the net gain in entropy of the working substance in one complete cycle ABCDA is

$$= Q_1/T_1 - Q_2/T_2$$

Since in a Carnot cycle $Q_1/T_1=Q_2/T_2$ hence the change in entropy in a reversible process is zero. i.e., in a reversible process the entropy remains same.

Now consider an irreversible process such as conduction or radiation in which heat is lost by a body at higher temperature T_1 and gained by the body at lower temperature $T_2(T_1>T_2)$ with the change in heat energy is given by Q

∴ Gain in entropy of the cold body = Q/T_1
Loss in entropy of the hot body = Q/T_2
Hence the net change in entropy of the system = $Q/T_2 - Q/T_1$
As $T_1 > T_2$ the change in the entropy is +ve quantity, hence entropy increases in irreversible process.

Entropy and second law of thermodynamics

If a process occurs in a closed system, the entropy of the system increases for irreversible processes and remains constant for reversible processes. It never decreases.

Entropy and Third law of thermodynamics:

As the temperature tends to absolute zero, the entropy also tends to zero and the molecules of a substance or a system are in perfect order is called third law of thermodynamics.

Example: The molecules in a gaseous state are more free as compared to liquid state, hence entropy is more in case of gas than liquid state i.e., as temperature increases degree of disorderness increases.

Entropy and probability:

According to Boltzmann theory the entropy(S) of particular macrostate and multiplicity/ number of microstates(w) are related by a logarithmic function given by

$S = k \ln w$ Where k = Boltzmann constant

The multiplicity defines the probability of occurrence of the different systems by the molecules.

Example if N no. of molecules in a system can be distributed in two systems such that n_1 molecules in state 1 and n_2 molecules in state 2 then their multiplicity is given by

$$w = \frac{N!}{n_1! \times n_2!}$$

An example: Imagine a gas consisting of just 2 molecules. We want to consider whether the molecules are in the left or right half of the container. There are 3 macrostates: both molecules on the left, both on the right, and one on each side. There are 4 microstates: LL, RR, LR, RL.

Then the multiplicity is given by

1 macro state and 3 macro state $w = \frac{2!}{2! \times 0!} = 1$ then $S = 0 \because \ln w = 0$

2 macro state $w = \frac{2!}{1! \times 1!} = 2$ then $S = 9.5 \times 10^{-24}$ J/K

Since the total entropy of two systems is the sum of their separate entropies, however the probability of two independent systems is the product of their separate probability.

UNIT-2 Part-I (waves & oscillations)

Waves:- A wave motion is a disturbance of some kind which moves from one place to another by means of a medium.

Oscillation:- Oscillation is an effect expressible as a quantity that repeatidly & regularly fluctuate above and below some mean position.

I) **Transverse wave:-** wave motion in which particle of medium vibrate about their mean position at right angle to the direction of propagation.
Eg- Light waves.

II) **Longitudinal waves:-** wave motion in which wave vibrate about their mean position along the same line as propagation of wave.
Eg. Sound waves

Simple Harmonic Motion:- If the acceleration of a particle in a periodic motion is always directly proportional to its displacement from its equilibrium position.

Types of SHM:-

1) Linear ~~Harmonic~~ Simple Harmonic Motion:-
If the displacement of a particle executing SHM is linear is said to be linear simple harmonic motion.
Example- Simple Pendulum.

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Angular Simple Harmonic Motion:-

If the Displacement of a Particle Executing SHM is angular. Example- compound Pendulum.

Essential Conditions for SHM:-

1) If f be the linear acceleration and x be the displacement from Equilibrium Position.

The essential condition is -

$$\boxed{f \propto -x}$$

2) If α be the angular Momentum θ be the angular displacement then essential condition is-

$$\boxed{\alpha = -\theta}$$

Time Period:- The smallest time interval during which oscillation repeat itself is called time Period, denoted by T. Its Unit is seconds.

Frequency:- Number of oscillation that a body complete in one second is called frequency of Periodic Motion.

It is reciprocal of time Period T and is given by-

$$\boxed{n = \frac{1}{T}}$$

Unit - Hertz represented by Hz.

Amplitude:- Maximum displacement of a body from its Mean Position.

Phase :-> It is a physical quantity that express the instaneous position and direction of motion of an oscillating system.

Differential Equation of SHM & its solution:-

Let us consider a particle of mass m executing SHM along a straight line with x as displacement from the mean position at any time t. Then from the basic condition of SHM restoring force F will be proportional to displacement and will be directed opposite to it.

Therefore

$$F \propto -x$$

$$\boxed{F = -kx} \quad -(1)$$

k is proportionality constant known as force constant

If $a = \frac{d^2x}{dt^2}$ be the acceleration at any instant of time

From (1) $\quad F = -kx$

$$ma = -kx$$

$$m\frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \quad -(2)$$

Substituting $\frac{k}{m} = \omega^2$

$$\frac{d^2x}{dt^2} = -\omega^2 x \Rightarrow \boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0} \quad -(3)$$

Equation (1) & (2) are known as differential Equation of SHM.

Solution of Differential Equation:-

$$x = A\sin(\omega t + \delta)$$

This Equation gives the displacement of particle executing SHM at any instant of time.

$A \rightarrow$ Maximum Displacement of Particle

$$x = A\sin(\omega t + \delta) \quad -(1)$$

Replacing t by $\left(t + \frac{2\pi}{\omega}\right)$, then we have-

$$x = A\sin\left(\omega\left(t + \frac{2\pi}{\omega}\right) + \delta\right)$$

$$= A\sin(\omega t + \delta) \quad -(2)$$

(1) & (2) are same this shows that motion is repeated after an interval of $\frac{2\pi}{\omega}$. So this interval will be time period of SHM given by $T = \frac{2\pi}{\omega}$

Also we have-

$$\omega^2 = \frac{k}{m} \Rightarrow \omega = \sqrt{\frac{k}{m}}$$

Time Period:-

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

Frequency $\quad n = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

Phase:- The quantity $(\omega t + \delta)$ is known as phase of vibrating particle. If $t=0$ then $(\omega t + \delta) = \delta$, so that initial phase will be δ.

If the particle start from mean position, then $\delta = 0$ for a particle start from extreme position then $\boxed{\delta = \pi/2}$

Velocity & Acceleration:- For a particle executing SHM, expression for displacement is -

$$x = A \sin(\omega t + \delta)$$

Differentiating it wrt time we get -

$$V = \frac{dx}{dt} = A\omega \cos(\omega t + \delta) \quad -①$$

$$= A\omega \sqrt{1 - \sin^2(\omega t + \delta)}$$

$$= \omega \sqrt{A^2 - A^2 \sin^2(\omega t + \delta)}$$

$$V = \omega \sqrt{A^2 - x^2}$$

This is the expression for velocity of particle at any displacement x.

I) Maximum velocity is obtained by substituting

$$x = 0$$

$$\therefore \boxed{V_{max} = \omega A}$$

because $x = 0$ correspond to mean position, so the particle has maximum velocity when it is at mean position.

II) At extreme position

At extreme position $x = A$

on differentiating ① wrt t we get -

$$f = \frac{dv}{dt} = \frac{d}{dt}(A\cos(\omega t + \delta))$$

$$= A\omega^2 \sin(\omega t + \delta)$$

$$\boxed{f = -\omega^2 x} \quad (\text{using } x = A\sin(\omega t + \delta))$$

This is standard equation of SHM

$$\boxed{f_{max} = \omega^2 A}$$

$$\boxed{f_{min} = 0} \text{ at mean position}$$

Del operator:-

∇ operator is denoted by ∇ and is treated as a vector in Cartesian coordinates.

it is written as.

$$\vec{\nabla} = \hat{i}\frac{d}{dx} + \hat{j}\frac{d}{dy} + \hat{k}\frac{d}{dz}$$

Gradient of Scaler Field :-

Let $\phi(r)$ be a Scaler field $\vec{r}$ is the position of vector $(\vec{r} = x\hat{i} + y\hat{j} + z\hat{k})$

where (x, y, z) are the coordinates.

ϕ be a function of three coordinates.

then

$$\boxed{\text{grad}\,\phi = \nabla.\phi}$$

or $$\text{grad}\,\phi = \left(\hat{i}\frac{d}{dx} + \hat{j}\frac{d}{dy} + \hat{k}\frac{d}{dz}\right)\phi$$

$$\boxed{\text{grad}\,\phi = \left(\hat{i}\frac{d\phi}{dx} + \hat{j}\frac{d\phi}{dy} + \hat{k}\frac{d\phi}{dz}\right)}$$

Curl of a vector

Circulation of a closed field around a closed path. is given by curl of a vector

Mathematically

$$\boxed{\text{Curl of } \vec{F} = \nabla \times \vec{F}}$$

ie cross product between ∇ & $\vec{F}$

if $\boxed{\nabla \times \vec{F} = 0}$ then field is irrotational.

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Numericals problem

① If $r = x\hat{i} + y\hat{j} + z\hat{k}$ find $\nabla \cdot r$

$$\nabla \cdot r = \left(i\frac{d}{dx} + \hat{j}\frac{d}{dy} + \hat{k}\frac{d}{dz}\right)(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \left(\frac{dx}{dx} + \frac{dx}{dx} + \frac{dx}{x}\right)$$

$$= 1 + 1 + 1 = 3$$

② If $\phi = 4x^3y^2z^4$ find $\nabla\phi$

$$\nabla\phi = \left[\hat{i}\frac{d}{dx} + \hat{j}\frac{d}{dx} + \hat{k}\frac{d}{dz}\right](4x^3y^2z^4)$$

$$= \left[i\frac{d}{dx}(4x^3y^2z^4) + \hat{j}\left(\frac{d}{dx}(4x^3y^2z^4)\right) + k\frac{d}{dz}(4x^3y^2z^4)\right]$$

$$= 12x^2y^2z^4\,\hat{i} + 8x^3yz^4\,\hat{j} + 16x^3y^2z^3\,\hat{k}$$

Q3 If $A = 2xy + 2y + z^2$ find $\nabla \cdot A$

$$\nabla \cdot A = \left[\hat{i}\frac{d}{dx} + \hat{j}\frac{d}{dy} + k\frac{d}{dz}\right)(2xy + 2y + z^2)$$

$$= \hat{i}\frac{d}{dx}(2xy + 2y + z^2) + \hat{j}\left(\frac{d}{dy}(2xy + 2y + z^2)\right) + \hat{k}\left(\frac{d}{dz}(2xy + 2y + z^2)\right)$$

$$= \hat{i}(2y) + \hat{j}(2x + 2) + \hat{k}(2z)$$

Q Find the curl of a field $F_1 = x\hat{i} + y\hat{j} + z\hat{k}$

prove that it is rotational or irrotational.

Sol: Curl of a field $F_1 = \nabla \times F_1$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ d/dx & d/dy & d/dz \\ x & y & z \end{vmatrix}$$

where $\nabla = \hat{i}\frac{d}{dx} + \hat{j}\frac{d}{dy} + \hat{k}\frac{d}{dz}$

$$= \hat{i}\left(\frac{dz}{dy} - \frac{dy}{dz}\right) + \hat{j}\left(\frac{dz}{dx} - \frac{dx}{dz}\right) + k\left(\frac{dy}{dx} - \frac{dx}{dy}\right)$$

$\Rightarrow \nabla \times F_1 \neq 0$

So field is rotational.

Gauss Divergence Theorm:-

It States that, the volume Integral of the divergence of Vector field A taken over any volume V bounded by a Closed surface S is equal to Surface integral of A taken over the surface S.

Mathematically

$$\iiint_V \operatorname{div} A \, dv = \iint_S A \cdot ds$$

Stokes Theorm:-

It State that the surface integral of a curl of a vector field A taken over any surface S is equal to the line integral of A around a closed curve.

$$\iint_S (\operatorname{curl} A) \, ds = \oint A \cdot dl$$

OR

$$\iint_S (\nabla \times A) \, ds = \oint A \cdot dl.$$

Physical significance & derivation of Maxwell's Equations

To derive Maxwell Equ^n we need to study following terms -

1) **Gauss law of Electrostatics:-**

Gauss law states that Electric flux through any closed surface is Equal to net charge Enclosed by the surface divided by Permitivity of vaccum.

$$\phi = \frac{Q}{\epsilon_0} \text{ or } \frac{q}{\epsilon_0}$$

Gauss law of Electrostatics (Integral Form)

The number of lines of force passing through a small area element ds is given by.

$$d\phi = E \cdot ds = E ds \cos\theta$$

This is known as Electric flux of the field over the elementary surface ds.

For a closed surface flux of field is given by.

$$\phi = \oiint E \cdot ds$$

In Integral form

Total outward flux of Electric field over a closed surface is equal to $\frac{1}{\epsilon_0}$ ~~the~~ times the total Net charge contained in a volume enclosed by the surface

$$\boxed{\oiint E \cdot ds = q/\epsilon_0}$$

E - Electric field Intensity ds - Surface Element.

Gauss law of Electrostatics: (Differential Form)

Let E be the Electric Field ~~of~~ at the centre of an elementary area dS on the closed surface S. Let ρ be the volume charge density of volume Enclosed by surface S. ρdV may be considered as a point charge contributing electric flux to the elementary area dS.

$$\Rightarrow q = \rho dV$$

Then, Total flux of whole surface -

$$\oint_S E \cdot dS = \frac{1}{\epsilon_0} \int \rho dV \quad -①$$

According to divergence theorm we have -

$$\oint_S E \cdot dS = \oint_V (\nabla \cdot E) dV \quad -②$$

Equating ①, ② we have

$$\oint_V (\nabla \cdot E) dV = \frac{1}{\epsilon_0} \int \rho dV$$

$$\oint_V (\nabla \cdot E) dV = \int_V \frac{\rho}{\epsilon_0} dV \quad -③$$

Since dV is arbitrary volume Element ③ is valid for any volume

Hence $\boxed{\nabla \cdot E = \frac{\rho}{\epsilon_0}}$

OR $\quad \text{div } E = \frac{\rho}{\epsilon_0}$

Gauss Law in Magnetostatics:-

I) Differential form:-

Since in Magnetic field lines are continuous the magnetic field Entering any Region is equal to Magnetic flux leaving it. so net flux over a volume is zero

Mathematically $\boxed{\nabla \cdot B = 0}$

This is Differential form of gauss law.

II) Integral form:-

Net flux over a closed surface is -

$$\phi = \oint B \cdot ds$$

Applying divergence theorem we have -

$$\oint_S B \cdot ds = \int_V (\nabla \cdot B) \, dv = 0$$

$$\boxed{\oint B \cdot ds = 0}$$

This is integral form of gauss law.

Faraday Law:- I) Magnetic flux through a closed loop of conducting wire varies in time with EMF.

$$\oint E \cdot dl = -\frac{d\phi}{dt} = \oint_S \cancel{\frac{d(B \cdot ds)}{dt}}$$

$$= \oint_S \frac{dB \cdot d\vec{s}}{dt}$$

ii) Negative time rate of variation of Magnetic flux linked with circuit is equal to EMF induced within it.

$$\boxed{\nabla \times E = -\frac{dB}{dt}}$$

Ampere's circuital law:-

This law state that line integral of Magnetic field (B) around any closed path or loop is equal to μ_0 times the total current enclosed by the loop.

$$\boxed{\oint B \cdot dl = \mu_0 I}$$

Maxwell Equation (Differential Form)

First

I) $\nabla \cdot D = \rho$ (Gauss law of Electrostatics)

OR

$\nabla \cdot E = \rho / \epsilon_0$ (Using $D = \epsilon_0 E$)

Second

II) $\nabla \cdot B = 0$ (Gauss law of Magnetostatics)

Third

III) $\nabla \times E = -\frac{dB}{dt}$ (Faraday law)

Fourth

IV) $\nabla \times B = \mu_0 \left(J + \frac{dD}{dt} \right)$ (Modified Ampere Circuital law)

OR

$\nabla \times H = \left(J + \frac{dD}{dt} \right)$ (Using $B = \mu_0 H$)

Maxwell Equation (Integral Form)

First

I) $\oint E \cdot ds = \frac{q}{\epsilon_0}$

Second

II) $\oint B \cdot ds = 0$

Third

III) $\oint E \cdot dl = -\frac{d\Phi_B}{dt}$

Fourth

IV) $\oint B \cdot dl = \mu_0 \left(J + \frac{dD}{dt} \right) \cdot ds$

where

D- Displacement or Electric Displacement (coulomb/m2)

ρ- Charge Density (coulom/m3)

B- Magnetic Induction (wb/m2) or flux density

H- Magnetic Field Intensity (Amp/m)

J- Current Density (I/A) (current/Ampere)

First Maxwell Equation:-

$$\nabla \cdot E = \rho/\epsilon_0$$

Physical significance:-

→ It is based on gauss law of Electrostatics

→ Net Electric flux through a closed surface is Equal to $\frac{1}{\epsilon_0}$ the total charge Enclosed by the surface.

$$\nabla \cdot E = \frac{q}{\epsilon_0} \qquad \oint_S E \cdot ds = \frac{1}{\epsilon_0}\int_V q \cdot dv \quad (1)$$

(Differential form) (Integral form)

→ It relate Electric flux with charge.

→ charge acts a a source or sink for the line of Electric force

Derivation:- Consider a surface bounded by a volume V in a medium having charge density as ρ.

then

$$q = \int_V \rho dV$$

using (1) Gauss law of Electrostatics we have

$$\oint E \cdot ds = \frac{q}{\epsilon_0}$$

Subustituting $q = \int_V \rho dV$

then

$$\oint E \cdot ds = \frac{1}{\epsilon_0}\int \rho dV \quad -(2)$$

From Gauss theorm we have.

$$\oint_S E \cdot ds = \int_V (\nabla \cdot E)\, dv \quad -③$$

(Note Gauss Divergence theorm converts Surface Integral to volume Integral)

Equating ②, ③ we get.

$$\int_V (\nabla \cdot E)\, dv = \frac{1}{\epsilon_0} \int_V \rho\, dv$$

$$\int_V \left(\nabla \cdot E - \frac{\rho}{\epsilon_0} \right) dv = 0$$

$$\Rightarrow \boxed{\nabla \cdot E = \rho / \epsilon_0}$$

Maxwell ① Equation.

~~Maxwell's Second Equation:-~~

Maxwell's Second Equation:-

$$\nabla \cdot B = 0$$

Significance:-

- It is based on Gauss law of Magnetostatics.
- It states that magnetic flux through any closed surface is zero

Differential form

$$\nabla \cdot B = 0$$

Integral form

$$\oint B \cdot ds = 0$$

- Time Independent Equation
- Magnetic flux is zero
- According to this Equation, isolated magnetic poles don't exist.

Derivation:- We have by Gauss law of Magnetostatics

$$\oint B \cdot ds = 0 \quad -①$$

By Gauss divergence theorm we have.

$$\oint_S B \cdot ds = \int_V (\nabla \cdot B)\, dv \quad -②$$

(Note:- By Gauss divergence theorm we have Surface Integral Equal to volume Integral)

Equating ① , ②

$$\int_V (\nabla \cdot B)\, dv = 0$$

$$\boxed{\nabla \cdot B = 0} \rightarrow \text{Maxwell Second Equation}$$

Maxwell's Third Equation:-

$$\nabla \times E = -\frac{dB}{dt}$$

Significance:-

⇒ This Equation represent Faraday law of Electro-Magnetic Induction.

Differential form

$$\nabla \times E = -\frac{dB}{dt}$$

Integral form

$$\oint E \cdot dl = -\frac{d\phi}{dt}$$

→ It is time dependent Equation.

→ Relates space variation of E with variation of B

→ Time variation of Magnetic field generates Electric field.

→ Negative sign Justify Lenz law.

Derivation:- we have by Faraday law.

$$\oint_c E \cdot dl = -\frac{d\phi}{dt}$$

$$= -\frac{d(B \cdot ds)}{dt} \qquad (\phi = B \cdot ds) \quad -(1)$$

By Stokes theorm we have.

$$\oint_c E \cdot dl = \int_S (\nabla \times E) \cdot ds \quad -(2)$$

Equating (1) (2)

$$\oint_S (\nabla \times E)\, ds = -\frac{d}{dt}\int_S B \cdot ds$$

$$\oint_S (\nabla \times E)\, ds = - \int \frac{dB}{dt} \cdot ds$$

$$\Rightarrow \boxed{\nabla \times E = -\frac{dB}{dt}}$$ — Maxwell Third Eqn

Maxwell fourth Equation

$$\text{curl } H = J + \frac{dD}{dt}$$

Significance

→ It represent Modified form of Amper's law.

→ It is time dependent Equation

→ It shows that Magnetic field can be generated by Current density vector and time variation

Differential form

$$\nabla \times B = \mu_0 \left(J + \frac{dD}{dt} \right)$$

Integral form

$$\oint_C B \cdot dl = \mu_0 \int_S \left(J + \frac{dD}{dt} \right)$$

Derivation:- By Amper's law we have.

$$\oint B \cdot dl = \mu_0 I$$

$$\oint (\cancel{\mu_0} H) dl = \cancel{\mu_0} I$$

$$\oint H \cdot dl = I$$

$$I = \int_S J \cdot ds$$

$$\Rightarrow \int_C H \cdot dl = \int_S J \cdot ds \quad -(1)$$

By Stokes theorm we have.

$$\int_C H \cdot dl = \int_S (\nabla \times H) ds \quad -(2)$$

Equating (1), (2)

$$\int_S (\nabla \times H) ds = \int_S J \cdot ds$$

$$\int_S (\nabla \times H - J) ds = 0$$

$$\Rightarrow \nabla \times H - J = 0 \quad \Rightarrow \boxed{\text{Curl } H = J} \quad -(3)$$

using The equation of time varying field.

$$\text{div } J + \frac{d\rho}{dt} = 0$$

$$\text{div } J = -\frac{d\rho}{dt} \quad -(4)$$

Maxwell added a current density J_d to original current density J ie

$$C = J + J_D$$

using (3) $\quad \text{Curl } H = J + J_D \quad -(6)$

$$\text{div}(\text{curl } \vec{H}) = \text{div}(J + J_D)$$

$$\text{div}(J + J_D) = 0$$

$$(\text{div}(\text{curl } \vec{H}) = 0 \text{ for time varying field})$$

$$\Rightarrow \text{div } J + \text{div } J_D = 0$$

$$\text{div } J_D = -\text{div } J$$

using (4) $\quad \text{div } J_D = +d\rho/dt \quad -(5)$

But $\boxed{\text{div } D = \rho}$ By maxwell (1) equ

$$\text{div } J_D = \frac{d}{dt}(\text{div } D)$$

$$= \text{div}\left(\frac{dD}{dt}\right)$$

$$\boxed{J_D = \frac{dD}{dt}}$$

using (6) $\quad \text{Curl } H = J + J_D$

$$\boxed{\text{Curl } H = J + \frac{dD}{dt}}$$

Maxwell's (4) Equation

Conversion of Maxwell Equation from Differential to Integral Form!

① **First Equation:-**

$$\text{div } E = \rho/\epsilon_0 \quad -①$$

Integrating above Equation over the volume V then-

$$\int_V \text{div } \vec{E} \, dv = \int_V (\rho/\epsilon_0) \, dv$$

Applying divergence theorm $\oint_S E\cdot ds = \int_V \text{div } E \, dv$

Using ①
$$\oint_S E\cdot ds = \frac{1}{\epsilon_0} \int_V \rho \, dv$$

$$\boxed{\oint_S E\cdot ds = \frac{q}{\epsilon_0}}$$ which is simple gauss law.

② **Second Equation.**

$$\text{div } B = 0 \quad \text{or} \quad \nabla\cdot B = 0$$

Taking integral both sides.

$$\oint B\cdot d\underset{\sim}{S} = 0$$

ie Magnetic flux through a closed surface is always zero.

This implies absence of Magnetic Monopoles.

(3) Third Equation we have -

$$\nabla \times E = -dB/dt$$

Taking Integral both Sides.

$$\oint_S (\nabla \times E)\, ds = -\oint_S \frac{dB}{dt} \cdot ds \quad -(1)$$

Applying Stoke's theorm.

$$\int_C E.dl = \oint_S (\nabla \times E)\, ds \quad -(2)$$

Substituting in (1)

$$\oint E.dl = -\int \frac{dB}{dt} . ds$$

$$= -\frac{d}{dt}\int B.S = -\frac{d\phi_B}{dt}$$

(4) fourth Equation.

$$\nabla \times B = \mu_0 \left(J + \frac{dD}{dt} \right)$$

On Integrating both sides.

$$\oint (\nabla \times B) . ds = \mu_0 \oint_S \left(J + \frac{dD}{dt} \right) . ds$$

Using Stokes theorm we get.

$$\oint_S (\nabla \times B)\, ds = \mu_0 \oint_S \left(J + \frac{dD}{dt} \right) . ds$$

$$\boxed{\oint B.dl = \mu_0 \int_S \left(J + \frac{dD}{dt} \right) . ds}$$

Integral fom of Fourth Equation

Electromagnetic Wave Equation:- (Part 3)

I) In Free space

II) In Dielectric Medium

III) In conducting Medium

Imp

I) Propagation of Electromagnetic Wave Equation in Free Space

It can be derived by using Maxwell's Equation.

Maxwell's Equation in general form are given by.

$$\left.\begin{aligned} &\text{I)}\quad \nabla\cdot E = \rho/\epsilon_0 \\ &\text{II)}\quad \nabla\cdot B = 0 \\ &\text{III)}\quad \nabla\times E = -\frac{dB}{dt} \\ &\text{IV)}\quad \nabla\times B = \mu_0\left(J+\frac{dD}{dt}\right) \end{aligned}\right\} \quad \text{— (A)}$$

Now in vaccum (free space) we have.

$$\boxed{D = \epsilon_0 E}\ , \quad \boxed{B = \mu_0 H}$$

Conductivity $\boxed{\sigma = 0}$ (i.e. vacuum is non-conducting)

No free electron are there i.e. $\boxed{J=0}$, $\boxed{\rho=0}$

Using these conditions in (1) we get.

$$\left.\begin{aligned} &\text{I)}\quad \nabla\cdot E = 0 \\ &\text{II)}\quad \nabla\cdot H = 0 \\ &\text{III)}\quad \nabla\times E = \cancel{-\frac{dB}{dt}} -\mu_0 \frac{dH}{dt} \\ &\text{IV)}\quad \nabla\times H = \epsilon_0 \frac{dE}{dt} \end{aligned}\right\} \quad \text{— (B)}$$

Now

I) Equation for Electric field vector -

Taking curl on both side of (III) of (B)

$$\nabla \times (\nabla \times E) = \nabla \times \left(-\mu_0 \frac{dH}{dt}\right)$$

$$= -\mu_0 \frac{d}{dt} (\nabla \times H)$$

or $\nabla \cdot (\nabla \cdot E) - \nabla^2 E = -\mu_0 \frac{d}{dt} (\nabla \times H)$

(using $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$)

Using (I, IV) of B

$$0 - \nabla^2 E = -\mu_0 \epsilon_0 \frac{d^2 E}{dt^2}$$

$$\Rightarrow \boxed{\nabla^2 E = \mu_0 \epsilon_0 \frac{d^2 E}{dt^2}} \quad -(1)$$

This is general Equation of Electromagnetic waves in term of Electric field vector $\vec{E}$.

II) Equation for Magnetic field vector.

Taking curl on both side of Equation (IV) of (B) we have.

$$\nabla \times (\nabla \times H) = \epsilon_0 \left(\nabla \times \frac{dE}{dt}\right)$$

$$\Rightarrow \nabla \times (\nabla \times H) = \epsilon_0 \left(\frac{d}{dt} (\nabla \times E)\right)$$

or $\nabla (\nabla \cdot H) - \nabla^2 H = \epsilon_0 \left(\frac{d}{dt} (\nabla \times E)\right)$

Using II, III of (B)

$$0 - \nabla^2 H = -\mu_0 \epsilon_0 \frac{d^2 H}{dt^2}$$

$$\Rightarrow \boxed{\nabla^2 H = \mu_0 \epsilon_0 \frac{d^2 H}{dt^2}} \quad -(2)$$

This is Electromagnetic wave Equation in term of magnetic field.

Wave velocity:- The Electromagnetic wave Equation for E & H (Electric & Magnetic vectors) are written as-

$$\nabla^2 E - \mu_0 \epsilon_0 \frac{d^2 E}{dt^2} = 0$$

$$\nabla^2 H - \mu_0 \epsilon_0 \frac{d^2 H}{dt^2} = 0$$

These Equation can be written by using x-coordinates-

$$\frac{d^2 E_x}{dx^2} - \mu_0 \epsilon_0 \frac{d^2 E_x}{dt^2} = 0 \quad - (3)$$

$$\frac{d^2 H_x}{dx^2} - \mu_0 \epsilon_0 \frac{d^2 H_x}{dt^2} = 0 \quad - (4)$$

These Equations are of type-

$$\frac{d^2 y}{dx^2} - \frac{1}{c^2} \frac{d^2 y}{dt^2} = 0 \quad - (5)$$

Equation (5) is general Equation of wave travelling with velocity c.

on comparing (3), (4), (5) we conclude that E & H travel in free space in term of wave with velocity c given by

$$\frac{1}{c^2} = \mu_0 \epsilon_0$$

$$\Rightarrow \quad c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

But $\mu_0 = 4\pi \times 10^{-7} NA^{-2}$, $\epsilon_0 = 8.85 \times 10^{-12} C^2 N^{-1} M^{-2}$

$$\Rightarrow \quad c = \frac{1}{\sqrt{4\pi \times 10^{-7} NA^{-2} \times 8.85 \times 10^{-12} C^2 N^{-1} M^{-2}}}$$

$$= 2.99792 \times 10^8 MS^{-1} = 3 \times 10^8 MS^{-1}$$

ie velocity of Electromagnetic wave in free space equal to velocity of light

② Electromagnetic Wave Equation in Dielectric :-

Maxwell's Equation in General form is given by -

I) $\nabla \cdot E = \rho / \epsilon_0$

II) $\nabla \cdot B = 0$

III) $\nabla \times E = -dB/dt$

IV) $\nabla \times B = \mu_0 \left(J + \frac{dD}{dt} \right)$

For dielectric conductivity $\boxed{\sigma = 0}$ and $\boxed{J = \sigma E = 0}$

When Dielectric is not charged then $\boxed{\rho = 0}$

Then -

I) $\nabla \cdot E = 0$

II) $\nabla \cdot H = 0$

III) $\nabla \times E = -\mu_0 \frac{dH}{dt}$

IV) $\nabla \times H = \epsilon_0 \frac{dE}{dt}$

Now Replacing μ_0, ϵ_0 by μ & ϵ respectively in Equation ③ & ④ we get.

$$\nabla^2 E = -\mu\epsilon \frac{d^2E}{dt^2}$$

$$\nabla^2 H = -\mu\epsilon \frac{d^2H}{dt^2}$$

or
$$\frac{d^2E_x}{dx^2} - \mu\epsilon \frac{d^2E_x}{dt^2} = 0$$

$$\frac{d^2H_x}{dx^2} - \mu\epsilon \frac{d^2H_x}{dt^2} = 0$$

Comparing with general wave Equation.

$$\frac{d^2y}{dx^2} - \frac{1}{c^2}\frac{d^2y}{dt^2} = 0$$

$\Rightarrow \frac{1}{c^2} = \mu\epsilon \Rightarrow \boxed{c = \frac{1}{\sqrt{\mu\epsilon}}}$ - wave velocity

Electromagnetic wave Equation in conducting Medium:-

Maxwell Equation in conducting Medium with permeability μ, Permitivity ϵ and conductivity σ can be written as-

1) $\nabla \cdot E = 0$ — (1)

II) $\nabla \cdot H = 0$ — (2)

III) $\nabla \times E = -\mu \frac{dH}{dt}$ — (3)

IV) $\nabla \times H = J + \epsilon \frac{dE}{dt}$

$= \sigma E + \epsilon \frac{dE}{dt}$ — (4)

Taking Curl of Equation (3) we have-

$$\nabla \times (\nabla \times E) = \nabla \times \left(-\mu \frac{dH}{dt}\right)$$

$$= -\mu \left(\nabla \times \frac{dH}{dt}\right)$$

$$= -\mu \left(\frac{d}{dt} (\nabla \times H)\right)$$

$$= -\mu \left(\frac{d}{dt} \left(\sigma E + \epsilon \frac{dE}{dt}\right)\right) \quad (\text{using } (4))$$

$$\nabla \times (\nabla \times E) = -\mu \sigma \frac{dE}{dt} - \mu \epsilon \frac{d^2E}{dt^2}$$

$$\Rightarrow \nabla \cdot (\nabla \cdot E) - \nabla^2 E = -\mu \sigma \frac{dE}{dt} - \mu \epsilon \frac{d^2E}{dt^2}$$

$$0 - \nabla^2 E = -\mu \sigma \frac{dE}{dt} - \mu \epsilon \frac{d^2E}{dt^2}$$

$$\nabla^2 E = \mu \sigma \frac{dE}{dt} + \mu \epsilon \frac{d^2E}{dt^2}$$

In case of Non- Conducting Medium $\boxed{\sigma = 0}$ then

$$\boxed{\nabla^2 E = \mu \epsilon \frac{d^2E}{dt^2}}$$

Now taking curl of Equation (4) we obtain

$$\nabla \times (\nabla \times H) = \nabla \times \left(\sigma E + \epsilon \frac{dE}{dt}\right)$$

$$= \sigma (\nabla \times E) + \epsilon \frac{d(\nabla \times E)}{dt}$$

(Using (3)) $$= \sigma \left(-\mu \frac{dH}{dt}\right) + \epsilon \frac{d}{dt}\left(-\mu \frac{dH}{dt}\right)$$

$$-\nabla^2 H = -\mu \sigma \frac{dH}{dt} - \mu \epsilon \frac{d^2 H}{dt^2}$$

$$\nabla^2 H = \mu \sigma \frac{dH}{dt} + \mu \epsilon \frac{d^2 H}{dt^2}$$

If Medium is non-conducting $\boxed{\sigma = 0}$

$$\Rightarrow \boxed{\nabla^2 H = \mu \epsilon \frac{d^2 H}{dt^2}}$$

Solution of wave Equation:-

$$E(r,t) = E_0 e^{i(k \cdot r - \omega t)}$$

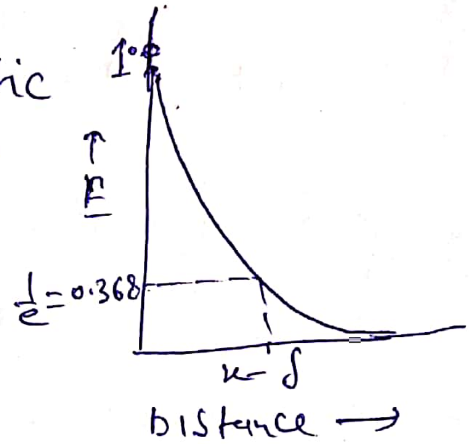
$$H(r,t) = H_0 e^{i(k \cdot r - \omega t)}$$

These two wave Equation Represent solution of wave Equation

Skin depth (Depth of Penetration)

It may be defined as the depth in which strength of Electric field associated with the Electromagnetic waves reduce to $1/e$ times to its initial value.

The Amplitude of strength of Electric field of an Electromagnetic waves decrease by a factor $e^{-\alpha x}$. where α is attenuation constant.
If depth of Penetration is represented by δ then-



$$x = \delta$$

or $$\delta = \frac{1}{\alpha}$$

Thus skin depth or Penetration depth

$$\delta = \frac{1}{\text{Attenuation Constant}}$$

Hence Reciprocal of Attenuation constant is called skin depth or Penetration depth.
where Attenuation constant is given by -

$$\alpha = \omega\left(\frac{\mu\epsilon}{2}\left(\sqrt{\left(1+\frac{\sigma^2}{\omega^2\epsilon^2}\right)}-1\right)\right)^{1/2}$$

Skin Depth $$\delta = \frac{1}{\omega\left(\frac{\mu\epsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}\right)-1\right)^{1/2}}$$

POYNTING THEORM:- / Imp

The rate of Energy transport per unit area, is called Poynting vector. It is also termed as instaneous energy flux density and is represented by S or P.

$$\boxed{\vec{S} = \vec{E} \times \vec{H}}$$

$\vec{S}$ is perpendicular to both $\vec{E}$ and $\vec{H}$.

Unit - W/m^2

Derivation:- We can calculate the energy density carried by Electromagnetic waves with the help of Maxwell's Equation given below-

i) $\nabla \cdot \vec{B} = 0$ — (1)

ii) $\nabla \cdot \vec{B} = 0$ — (2)

iii) $\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$ — (3)

iv) $\nabla \times \vec{H} = \vec{J} + \frac{d\vec{D}}{dt}$ — (4)

Now Taking (.) Dot product of ~~(3) with~~ H with (3) and Dot Product of E with (4)

$$\vec{H} \cdot (\nabla \times \vec{E}) = -\mu \vec{H} \cdot \frac{d\vec{H}}{dt} \quad — (5) \quad (B = \mu \vec{H})$$

$$\vec{E} \cdot (\nabla \times \vec{H}) = \vec{E} \cdot \vec{J} + \epsilon \vec{E} \frac{d\vec{E}}{dt} \quad — (6) \quad (D = \epsilon \vec{E})$$

Substracting Equation (5) from Equation (6)

$$\Rightarrow \vec{E} \cdot (\vec{\nabla} \times \vec{H}) - \vec{H} (\vec{\nabla} \times \vec{E}) = \vec{E} \cdot \vec{J} + \epsilon \vec{E} \frac{d\vec{E}}{dt} - \left(-\mu \vec{H} \frac{d\vec{H}}{dt}\right)$$

Since $\vec{A} \cdot (\vec{\nabla} \times \vec{B}) - \vec{B} \cdot (\vec{\nabla} \times \vec{A}) = \vec{\nabla} \cdot (\vec{B} \times \vec{A}) = -\vec{\nabla} (\vec{A} \times \vec{B})$

$\cancel{-\nabla\cdot(E\times H) = J\cdot E + \frac{1}{2}\epsilon\frac{d^2E}{dt^2}}$

$$\Rightarrow -\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{J}\cdot\vec{E} + \frac{1}{2}\epsilon\frac{dE^2}{dt} + \frac{1}{2}\mu\frac{dH^2}{dt}$$

using
$$\left(\because \epsilon\vec{E}\frac{dE}{dt} = \frac{1}{2}\epsilon\frac{dE^2}{dt} \quad \mu\vec{H}\frac{dH}{dt} = \frac{1}{2}\mu\frac{dH^2}{dt} \right\}$$

$$\Rightarrow -\vec{\nabla}\cdot\vec{S} = \vec{J}\cdot\vec{E} + \frac{1}{2}\frac{d}{dt}(\epsilon E^2 + \mu H^2)$$

or
$$\vec{J}\cdot\vec{E} + \frac{d}{dt}\left(\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2}\right) + \vec{\nabla}\cdot\vec{S} = 0$$

Taking volume Integral over the volume V enclosed by the surface S,

$$\int_V \vec{\nabla}\cdot\vec{S}\,dv + \frac{d}{dt}\int_V\left(\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2}\right)dv = -\int_V \vec{J}\cdot\vec{E}\,dv$$

using Divergence Theorm

$$\boxed{\int_S \vec{S}\cdot d\vec{S} + \frac{d}{dt}\int_V\left(\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2}\right)dv = -\int_V \vec{J}\cdot\vec{E}\,dv}$$

→ $\int_S \vec{S}\cdot d\vec{S}$ → Represent rate of flow of Energy or Power flux.

→ $\frac{d}{dt}\int_V\left(\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2}\right)dv$ = Rate of change of total Energy

→ $\frac{\epsilon E^2}{2}$ & $\frac{\mu H^2}{2}$ represent Energy stored in Electric & Magnetic field.

$\int_V \vec{J}\cdot\vec{E}\,dv$ = Rate of work done by the Electromagnetic Field in displacing the charge within the volume V.

Hence

$$\vec{\nabla}\cdot\vec{S} + \vec{J}\cdot\vec{E} + \frac{d}{dt}\left(\frac{1}{2}\mu H^2\right) + \frac{d}{dt}\left(\frac{1}{2}\epsilon E^2\right) = 0$$

OR

$$\int_V \vec{\nabla}\cdot\vec{S}\,dv + \int_V \vec{J}\cdot\vec{E}\,dv + \frac{d}{dt}\int_V \left(\frac{\mu H^2}{2} + \frac{\epsilon E^2}{2}\right) dv = 0$$

Above Equation is known as Poynting Theorem or work- Energy Theorm.

According to Poynting theorm the power transformed into Electromagnetic field is Equal to the sum of the time rate of change of EM energy. within certain volume and the time rate of Energy Flowing out through the boundary surface. This is also known as Energy conservation law in Electromagnetism.

Example 3.11 *The electromagnetic wave intensity received on the surface of the earth from the sun is found to be 1.33 kW/m^2. Find the amplitude of electric field vector associated with sunlight as received on earth surface. Assume Sun's light to be monochromatic ($\lambda = 6000$ Å).*

[GGSIPU, Feb. 2012 (5 marks) ; Feb. 2008 (3 marks)]

Solution. The energy transported by an electromagnetic wave per unit area per second during propagation is represented by Poynting vector **S** as

$$\mathbf{S} = (\mathbf{E} \times \mathbf{H})$$

The energy flux per unit area per second is

$$|\mathbf{S}| = |\mathbf{E} \times \mathbf{H}| = EH \sin 90° = EH$$

The energy flux per unit area per second at the earth surface.

$$|\mathbf{S}| = 1.33 \text{ kW/m}^2 = 1.33 \times 10^3 \text{Jm}^{-2}\text{s}^{-1}$$

$$\therefore \quad |\mathbf{S}| = 1330 \text{ Jm}^{-2}\text{s}^{-1} \qquad \text{...}(i)$$

We know that $$Z_0 = \left|\frac{\mathbf{E}}{\mathbf{H}}\right| = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\frac{4\pi\times10^{-7}\ \text{Wb/Am}}{8.854\times10^{-12}\ \text{C}^2/\text{Nm}^2}} = 376.72\ \Omega$$

$$\frac{E}{H} = 376.72\ \Omega \qquad ...(ii)$$

Multiplying Eq. (*i*) and Eq. (*ii*), we get

$$EH\times\frac{E}{H} = 1330\times376.72$$

or $$E^2 = 501037.6, \quad E = 707.8\ \text{V/m}$$

Substituting this value in Eq. (*ii*)

$$H = \frac{707.8}{376.72} = 1.879\ \text{A/m}.$$

Therefore, the amplitudes of electric and magnetic fields of radiation are

$$E_0 = E\sqrt{2} = 707.8\sqrt{2} = 1000.8292 = 1000\ \text{V/m}$$

and $$H_0 = H\sqrt{2} = 1.879\sqrt{2} = 2.657\ \text{A.turn/m}.$$

Example 3.12 If the earth receives 2 Cal $min^{-1}cm^{-2}$ solar energy, what are the amplitudes of electric and magnetic field of radiation ? **[GGSIPU., June 2015 (5 Marks), May 2016 (4 marks)]**

Solution. As Poynting vector,

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} = EH\sin90° = EH$$

$$\text{Solar energy} = 2\ \text{Cal min}^{-1}\text{cm}^{-2}$$

$$= \frac{2\times4.18\times10^4}{60}\ \text{Jm}^{-2}\text{s}^{-1}$$

Both are energy flux per unit area per second

Hence $$EH = \frac{2\times4.18\times10^4}{60} \approx 1400$$

But $$\frac{E}{H} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\ \pi = 377$$

$\therefore$ $$EH\times\frac{E}{H} = 1400\times377$$

$$E = \sqrt{1400\times377} = 726.5\ \text{V/m}$$

Now, $$H = \frac{E}{377} = 1.927\ \text{A/m}$$

Amplitudes of electric and magnetic field of radiation are

$$E_0 = E\sqrt{2} = 1024.3\ \text{V/m}$$

$$H_0 = H\sqrt{2} = 2.717\ \text{A/m}$$

Example 3.13 *Calculate the Poynting vector at the surface of the sun. Given that the energy radiated per second is* 3.8×10^{26} *J and the radius of the sun is* 0.7×10^9 *m. Also calculate the amplitudes of electric and magnetic field vectors on the surface of the earth. The distance of the earth from the sun is* 0.15×10^{12} *m.*

[GGSIPU., May 2014 (6 Marks)]

Solution. **First case.** The Poynting vector at the surface of sun

$$|\mathbf{S}| = |\mathbf{E} \times \mathbf{H}| = EH \sin 90°$$

$$= \frac{P_0}{4\pi r^2} = \frac{3.8 \times 10^{26} \text{ J/s}}{4\pi \times (0.7 \times 10^9 \text{ m})^2}$$

$$= 6.17 \times 10^7 \text{ Jm}^{-2}\text{s}^{-1}$$

Second case. The Poynting vector at the surface of earth

$$EH = \frac{P_0}{4\pi r_1^2}$$

$$= \frac{3.8 \times 10^{26} \text{ J/s}}{4\pi \times (0.15 \times 10^{12} \text{ m})^2} = 1344.656 \text{ Jm}^{-2}\text{s}^{-1} \quad ...(i)$$

We also know $$\frac{E}{H} = 120\pi\Omega = 377\ \Omega \quad ...(ii)$$

From Eqs. (*i*) and (*ii*)

$$\frac{E}{H} \times EH = 1344.656 \times 377$$

and $$E^2 = \sqrt{1344.656 \times 377} = 711.99 \text{ V/m}$$

and from Eq. (*ii*) $$\frac{E}{H} = 377\ \Omega$$

$$H = \frac{E}{377\Omega} = \frac{711.9938}{377} \text{ A/m} = 1.888 \text{ A/m}$$

Amplitude of electric and magnetic field of radiation are

$$E_0 = E\sqrt{2} = 711.99 \times 1.414 = 1006.75 \text{ V/m}$$

and $$H_0 = H\sqrt{2} = 1.888 \times 1.414 = 2.67 \text{ A/m}$$